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Remarks on some fourth-order boundary value problems with non-monotone increasing eigenvalue sequences

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Abstract. By Z_2-index theory, the existence and multiplicity of solutions for some fourth-order boundary value problems

$$\begin{cases} u^{(4)} + au'' = \mu u + F_u(t,u), \ 0 < t < L, \\ u(0) = u(L) = u''(0) = u''(L) = 0 \end{cases}$$

at resonance are studied, where $a > 0$ and $\mu \in \mathbb{R}$ is an eigenvalue of the corresponding eigenvalue problem. The difficulty caused by the non-monotone eigenvalue sequence is handled concretely.

Keywords: fourth-order differential equation, boundary value problem, resonance, critical point, Z_2-index theory.

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1 Introduction

In order to study physical, chemical and biological systems, one has considered some fourth-order semilinear differential equation models, such as the extended Fisher–Kolmogorov equation [4], the Swift–Hohenberg equation [10], suspended beam equations [2], etc. In the present paper, we are concerned with the existence and multiplicity of solutions for boundary value problems of fourth-order differential equations

$$\begin{cases} u^{(4)} + au'' = \mu u + F_u(t,u), \quad 0 < t < L, \\ u(0) = u(L) = u''(0) = u''(L) = 0 \end{cases} \tag{1.1}$$

where $a > 0$, μ is a real parameter, $F(t,u) \in C^1([0,L] \times \mathbb{R}, \mathbb{R})$, $F_u(t,u)$ denotes the gradient of $F(t,u)$ with respect to the variable u. If $F(t,u)$ satisfies $\lim_{|u|\to\infty} F(t,u)/u^2 = 0$, we say $F(t,u)$ is sublinear at infinity.

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We first observe that the corresponding eigenvalue problem

$$\begin{cases} u^{(4)} + au'' = \lambda u, \\ u(0) = u(L) = u''(0) = u''(L) = 0 \end{cases} \tag{1.2}$$

has the eigenvalues

$$\lambda_k = \left(\frac{k\pi}{L}\right)^4 - a\left(\frac{k\pi}{L}\right)^2, \qquad k = 1, 2, 3, \dots \tag{1.3}$$

and the eigenfunctions

$$u_k = \sin\frac{k\pi t}{L}, \qquad k = 1, 2, \dots \tag{1.4}$$

One says (1.1) is resonant at infinity if $\mu = \lambda_k$ and $F(t,u)$ is sublinear at infinity.

On the other hand, for fixed $a > 0$, we can find that from (1.2)

(i) if L is extremely small such that $(\frac{\pi}{L})^2 > a$, then $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty$;

(ii) if L is suitable large such that $\frac{a}{2} < (\frac{\pi}{L})^2 < a$, then $\lambda_1 < 0 < \lambda_2 < \lambda_3 < \cdots \to \infty$;

(iii) if L is sufficiently large , then there exists two integers $\hat{k} \geq 2$ and $\hat{l}$ satisfying $(\frac{\hat{k}\pi}{L})^2 < \frac{a}{2}$ and $0 > \lambda_1 > \lambda_2 > \cdots > \lambda_{\hat{k}}$, $\lambda_{\hat{k}} < \lambda_{\hat{k}+1} < \cdots < \lambda_{\hat{k}+\hat{l}} < 0 < \lambda_{\hat{k}+\hat{l}+1} < \lambda_{\hat{k}+\hat{l}+2} \cdots \to \infty$.

We see easily that for the case (i) and (ii), the eigenvalue sequence $\{\lambda_k\}$ increases strictly. In the last two decades, much of the research in critical point theory has examined the existence and multiplicity of solutions of (1.1), so, we shall focus on the most complicated case (iii) with non-monotone increasing eigenvalue sequence. For the sake of simplicity, we assume that $\lambda_{k_1} \neq \lambda_{k_2}$ if $k_1 \neq k_2$ in case (iii) throughout this paper.

Up to now there have been a vast of literature about superlinear (1.1), namely, $F(t,u)$ satisfies $\lim_{|u|\to\infty} F(t,u)/u^2 = \infty$; we refer the reader to [1,5,6,11,12], and references therein. However, for the sublinear (1.1), only a few attempts have been done. In [9], Liu considered the existence of solutions with $L = 1$, $\mu = -a^2/4$. In [7], Han–Xu supposed $L = 1$, $a = \mu = 0$, $F(t,u) < \gamma|u|^2 + \beta$ $(0 < \gamma < \pi^2/2)$ and $m^4\pi^4 < f_u(t,0) < (m+1)^4\pi^4$ with $m \geq 1$ and proved the existence of three solutions. In [13], Yang–Zhang assumed $L = 1$, $a\pi^4 + \mu/\pi^2 < 1$, $uf(t,u) - 2F(t,u) \to \infty$ $(|u| \to \infty)$, and discussed the existence of two solutions at resonance, basing on combining the minimax methods and the Morse theory. In [8], Li–Wang–Xiao used the Clark theorem to prove the following result.

Theorem 1.1. *Consider the problem*

$$\begin{cases} u^{(4)}(t) - V_u(t,u(t)) = 0, \quad 0 < l < L, \\ u(0) = u(L) = u''(0) = u''(L) = 0. \end{cases} \tag{1.5}$$

Let $V(t,u) \in C^1([0,L] \times \mathbb{R}, \mathbb{R})$ satisfy

$(V1)$ *$V(t,u) = V(t,-u)$, $\forall t \in \mathbb{R}$, $u \in \mathbb{R}$;*

$(V2)$ *there exist $m > 0$, $b > 0$ such that $L < \frac{\pi}{\sqrt[4]{m}}$ and*

$$V(t,u) \leq \frac{m}{2}|u|^2 + b, \qquad t \in [0,L],\ u \in \mathbb{R}; \tag{1.6}$$

$(V3)$ *there exist $p \in \mathbb{N}$ and constants $M > 0$, $\rho > 0$ such that $M > m\rho^4$, $L > \rho\pi/\sqrt[4]{m}$ and*

$$V(t,u) \geq \frac{M}{2}|u|^2, \qquad \forall t \in [0,L],\ |u| \leq \rho\sqrt{p}. \tag{1.7}$$

Then, for each $L \in \left(\frac{\pi p}{\sqrt[4]{M}}, \frac{\pi}{\sqrt[4]{m}}\right)$, (1.5) has at least p distinct pairs $(u(t), -u(t))$ of solutions.

Motivated mainly by the above papers [7–9, 13], we obtain the existence and multiplicity results for (1.1) as stated in the following.

Theorem 1.2. *Suppose that there exists some integer k such that $1 \leq k \leq \hat{k}$, and $\mu = \lambda_k$. Assume that $F(t,u) \in C^1([0,L] \times \mathbb{R}, \mathbb{R})$ and $f(t,u) = F_u(t,u)$ satisfy that*

$(f1)$ *$F(t,u) = F(t,-u)$, $\forall t \in [0,L]$, $u \in \mathbb{R}$;*

$(f2)$ *there exist $K_1, K_2 > 0$ and $\alpha \in [0,1)$ such that*

$$|f(t,u)| \leq K_1|u|^\alpha + K_2, \qquad \forall t \in [0,L],\ u \in \mathbb{R}; \tag{1.8}$$

$(f3)$ *there exist an integer $\hat{p} > 1$, and $M > 0$, $p > 0$ such that $\lambda_{\hat{k}+\hat{l}+\hat{p}} < M + \lambda_{\hat{k}}$ and*

$$F(t,u) \geq \frac{M}{2}|u|^2, \qquad \forall t \in [0,L],\ |u| \leq p; \tag{1.9}$$

$(f4)$ $$\liminf_{u=c\sin\frac{j\pi}{L}, |c|\to\infty} \int_0^L F(t,u(t))dt/|c|^\alpha \to -\infty, \qquad \forall j \geq 1.$$

Then (1.1) possesses at least $(\hat{k}+\hat{l}+\hat{p}) - (\bar{k}+1)$ distinct pairs $(u(t), -u(t))$ of solutions, where, if $\lambda_k > \min\{\lambda_j\}_{j\geq 1}$, then $\bar{k}$ satisfies $\hat{k}+1 \leq k+\bar{k} \leq \hat{k}+\hat{l}$, and $\lambda_j < \lambda_k$ if and only if $k+1 \leq j \leq k+\bar{k}$ ($\bar{k}$ is unique); if $\lambda_k = \min\{\lambda_j\}_{j\geq 1}$, then $\bar{k} = 0$.

Corollary 1.3. *In Theorem 1.2, if the condition $\mu = \lambda_k$ is replaced by $\lambda_k < \mu < \lambda_{k+1}$, and $(f4)$ is omitted, then the same conclusion still holds.*

Remark 1.4. In Theorem 1.2 and Corollary 1.3, if the integer k is between $\hat{k}+1$ and $\hat{k}+\hat{l}$, or $k > \hat{k}+\hat{l}$, then the similar results are still true. However, the details of their proofs must be adapted. Evidently, one can also handle case (i) and (ii) by the same arguments used by us in (iii), moreover, the process seems easier than that of (iii). In addition, it is also not hard to see that Theorem 1.1 is equivalent to the special situation of $a = 0$, $\mu < \lambda_1$ in Corollary 1.3.

This paper is organized as follows. In Section 2, we will prove some lemmas. In Section 3, the proof of Theorem 1.2 shall be given by the following Z_2-type index theorem.

Theorem 1.5. *Let Y be a Banach space and the functional $\varphi \in C^1(Y, \mathbb{R})$ be even satisfying the Palais–Smale condition. Suppose that*

(i) *there exist a subspace V of Y with $V = r$ and $\delta > 0$ such that $\sup_{w\in V, \|x\|=\delta} \varphi(w) < \varphi(0)$;*

(ii) *there exists a closed subspace W of Y with $W = s < r$ such that $\inf_{w\in W} \varphi(w) > -\infty$.*

Then f possesses at least $r - s$ distinct pairs $(u, -u)$ of critical points.

For the convenience of the reader, let us recall that the functional φ is said to satisfy the Palais–Smale condition if any sequence $\{u_j\}$ in Y is such that $\varphi(u_j)$ is bounded, $\varphi'(u_j) \to 0$, possesses a convergent subsequence.

2 Preliminary

Let $X = H^2(0, L; \mathbb{R}) \cap H_0^1(0, L; \mathbb{R})$ be a Hilbert space with the inner product

$$(u, w) = \int_0^L [u''(t)w''(t) + u'(t)w'(t) + u(t)w(t)]dt, \qquad \forall u, w \in X, \tag{2.1}$$

and the corresponding norm

$$\|u\|_X = (u, u)^{\frac{1}{2}} = \left(\int_0^L \left[|u''|^2 + a|u'|^2 + |u|^2\right] dt\right)^{\frac{1}{2}}. \tag{2.2}$$

Setting

$$\|u\| = \left(\int_0^L |u''|^2 dt\right)^{\frac{1}{2}}, \tag{2.3}$$

we infer from the Poincaré inequality

$$\int_0^L u^2 dt \leq \frac{L^4}{\pi^4} \int_0^L (u'')^2 dt, \tag{2.4}$$

$$\begin{aligned} \int_0^L u^2 dt &= \int_0^L u' du = -\int_0^L uu'' dt \\ &\leq \frac{1}{2} \int_0^L (u^2 + (u'')^2) dt \\ &\leq \frac{1}{2}\left(\frac{L^4}{\pi^4} + 1\right) \int_0^L (u'')^2 dt, \end{aligned} \tag{2.5}$$

so $\|\cdot\|_X$ is equivalent with $\|\cdot\|$.

It is well known that solutions of (1.1) are exactly the critical points of the corresponding functional given by

$$I(u) = \frac{1}{2} \int_0^L \left[|u''(t)|^2 - a|u'(t)|^2 - \mu|u|^2\right] dt - \int_0^L F(t, u) dt \tag{2.6}$$

on X. Direct computation shows, for $\forall u, w \in X$,

$$I'(u)w = \int_0^L \left[u''(t)w''(t) - au'(t)w'(t) - \mu u(t)w(t)\right] dt - \int_0^L f(t, u)v dt. \tag{2.7}$$

Obviously, there is an orthogonal basis on $(X, \ \|\cdot\|)$ as follows

$$\left\{\sin\frac{\pi}{L}t, \ \sin\frac{2\pi}{L}t, \ \sin\frac{3\pi}{L}t, \dots\right\}. \tag{2.8}$$

Let $e_j = \sin\frac{j\pi}{L}, s_j = (\frac{j\pi}{L})^4, \ j \geq 1$, then

$$\int_0^L |e_j''(t)|^2 dt = \frac{L}{2}\left(\frac{j\pi}{L}\right)^4 = s_j \int_0^L |e_j(t)|^2 dt. \tag{2.9}$$

Define $v_j = \sqrt{\frac{2}{Ls_j}} e_j$, we have $\|v_j\| = 1$, and

$$\int_0^L \left[|v_j''(t)|^2 - a|v_j'(t)|^2\right] dt = \frac{2}{Ls_j} \int_0^L \left[|e_j''(t)|^2 - a|e_j'(t)|^2\right] dt = \frac{\lambda_j}{s_j}, \tag{2.10}$$

$$\int_0^L |v_j(t)|^2 dt = \frac{2}{Ls_j}\int_0^L |e_j(t)|^2 dt = \frac{2}{Ls_j}\cdot\frac{L}{2} = \frac{1}{s_j}. \tag{2.11}$$

From (2.10) and (2.11), we obtain

$$\int_0^L \left[|v_j''(t)|^2 - a|v_j'(t)|^2\right] dt = \lambda_j \int_0^L |v_j|^2 dt. \tag{2.12}$$

Lemma 2.1. *Under the assumptions of Theorem 1.2, there exists a norm $\|\cdot\|_*$ equivalent with $\|\cdot\|$ on X; X has an orthogonal decomposition $X = X^+ \oplus X^- \oplus X^0$, and the functional $I(u)$ in (2.6) is of the form*

$$I(u) = \frac{1}{2}\left(\|u^+\|_*^2 - \|u^-\|_*^2\right) - \int_0^L F(t,u)dt, \qquad \forall u \in X, \tag{2.13}$$

where $u = u^+ \oplus u^- \oplus u^0$, $u^+ \in X^+$, $u^- \in X^-$, $u^0 \in X^0$.

Proof. Consider two cases. Case (i): $\mu = \lambda_k > \min\{\lambda_j\}_{j\geq 1}$. For this, there exists a unique integer $\bar{k}$ such that $\hat{k}+1 \leq k+\bar{k} \leq \hat{k}+\hat{l}$, and $\lambda_j < \lambda_k$ if and only if $k+1 \leq j \leq k+\bar{k}$. Define

$$I_1(u) = \int_0^L \left[|u''(t)|^2 - a|u'(t)|^2 - \mu|u|^2\right] dt. \tag{2.14}$$

For $\forall u \in X, u = \sum_{j=1}^{\infty} \alpha_j v_j$, we get $\|u\|^2 = \sum_{j=1}^{\infty}\alpha_j^2$, and

$$\begin{aligned}
I_1(u) &= \int_0^L \left[\left|\left(\sum_{j=1}^{\infty}\alpha_j v_j\right)''\right|^2 - a\left|\left(\sum_{j=1}^{\infty}\alpha_j v_j\right)'\right|^2 - \lambda_k\left(\sum_{j=1}^{\infty}\alpha_j v_j\right)^2\right] dt \\
&= \sum_{j=1}^{\infty}\left[\alpha_j^2\int_0^L |v_j''|^2 dt - a\alpha_j^2\int_0^L |v_j'|^2 dt - \lambda_k\alpha_j^2\int_0^L |v_j|^2 dt\right] \\
&= \sum_{j=1}^{\infty}\left(\frac{\lambda_j-\lambda_k}{s_j}\right)\alpha_j^2 \\
&= \sum_{j=1}^{k-1}\left(\frac{\lambda_j-\lambda_k}{s_j}\right)\alpha_j^2 + \sum_{j=k+1}^{k+\bar{k}}\left(\frac{\lambda_j-\lambda_k}{S_j}\right)\alpha_j^2 + \sum_{j=k+\bar{k}+1}^{\infty}\left(\frac{\lambda_j-\lambda_k}{s_j}\right)\alpha_j^2.
\end{aligned} \tag{2.15}$$

We define

$$u^+ = \sum_{j=1}^{k-1}\alpha_j v_j + \sum_{j=k+\bar{k}+1}^{\infty}\alpha_j v_j, \qquad u^- = \sum_{j=k+1}^{k+\bar{k}} a_j v_j, \qquad u^0 = a_k v_k, \tag{2.16}$$

and

$$\begin{aligned}
X^+ &= Span\{v_j \mid 1 \leq j \leq k-1 \text{ or } j \geq k+\bar{k}+1\}, \\
X^- &= Span\{v_j \mid k+1 \leq j \leq k+\bar{k}\}, \\
X^0 &= Span\{v_k\},
\end{aligned} \tag{2.17}$$

then we derive

$$u = u^+ + u^- + u^0, \qquad u^+ \in X^+, \qquad u^- \in X^-, \qquad u^0 \in X^0, \qquad X = X^+ \oplus X^- \oplus X^0. \tag{2.18}$$

Moreover, one can estimate the terms in (2.15) as follows:

$$\begin{aligned}\sum_{j=1}^{k-1}\left(\frac{\lambda_j-\lambda_k}{s_j}\right)\alpha_j^2+\sum_{j=k+\bar{k}+1}^{\infty}\left(\frac{\lambda_j-\lambda_k}{s_j}\right)\alpha_j^2\\ \geq\left(\frac{\lambda_{k-1}-\lambda_k}{s_j}\right)\sum_{j=1}^{k-1}\alpha_j^2+\left(\frac{\lambda_{k+\bar{k}+1}-\lambda_k}{s_{k+\bar{k}+1}}\right)\sum_{j=k+\bar{k}+1}^{\infty}\alpha_j^2,\end{aligned}\tag{2.19}$$

$$\sum_{j=1}^{k-1}\left(\frac{\lambda_j-\lambda_k}{s_j}\right)\alpha_j^2+\sum_{j=k+\bar{k}+1}^{\infty}\left(\frac{\lambda_j-\lambda_k}{s_j}\right)\alpha_j^2\leq\left(\frac{\lambda_1-\lambda_k}{s_1}\right)\sum_{j=1}^{k-1}\alpha_j^2+\sum_{j=k+\bar{k}+1}^{\infty}\alpha_j^2,\tag{2.20}$$

and

$$\sum_{j=k+1}^{k+\bar{k}}\left(\frac{\lambda_k-\lambda_j}{s_j}\right)\alpha_j^2\geq\frac{\mu_k^1}{s_{k+\bar{k}}}\sum_{j=k+1}^{k+\bar{k}}\alpha_j^2,\tag{2.21}$$

$$\sum_{j=k+1}^{k+\bar{k}}\left(\frac{\lambda_k-\lambda_j}{s_j}\right)\alpha_j^2\leq\frac{\mu_k^2}{s_{k+1}}\sum_{j=k+1}^{k+\bar{k}}\alpha_j^2,\tag{2.22}$$

with

$$\mu_k^1=\min_{k+1\leq j\leq k+\bar{k}}\{\lambda_k-\lambda_j\}>0,\qquad \mu_k^2=\max_{k+1\leq j\leq k+\bar{k}}\{\lambda_k-\lambda_j\}>0.\tag{2.23}$$

Combining (2.19)–(2.20) with (2.21)–(2.22), we can define a new norm $\|\cdot\|_*$ on X by

$$\|u\|_*^2=\sum_{j=1}^{k-1}\left(\frac{\lambda_j-\lambda_k}{s_j}\right)\alpha_j^2+\sum_{j=k+1}^{k+\bar{k}}\left(\frac{\lambda_j-\lambda_k}{S_j}\right)\alpha_j^2+\sum_{j=k+\bar{k}+1}^{\infty}\left(\frac{\lambda_j-\lambda_k}{s_j}\right)\alpha_j^2+\alpha_k^2,\tag{2.24}$$

equivalent with $\|\cdot\|$. If $u=\sum_{j=1}^{\infty}\alpha_j v_j, w=\sum_{j=1}^{\infty}\beta_j v_j\in X$, then the inner product corresponding to $\|\cdot\|_*$ is

$$\begin{aligned}\langle u,\ w\rangle_*=\sum_{j=1}^{k-1}\left(\frac{\lambda_j-\lambda_k}{s_j}\right)\alpha_j\beta_j+\sum_{j=k+\bar{k}+1}^{\infty}\left(\frac{\lambda_j-\lambda_k}{s_j}\right)\alpha_j\beta_j\\ +\sum_{j=k+1}^{k+\bar{k}}\left(\frac{\lambda_k-\lambda_j}{s_j}\right)\alpha_j\beta_j+\alpha_k\beta_k.\end{aligned}\tag{2.25}$$

From (2.24) one gets

$$\begin{aligned}\|u^+\|_*^2&=\sum_{j=1}^{k-1}\left(\frac{\lambda_j-\lambda_k}{s_j}\right)\alpha_j^2+\sum_{j=k+\bar{k}+1}^{\infty}\left(\frac{\lambda_j-\lambda_k}{s_j}\right)\alpha_j^2,\\ \|u^-\|_*^2&=\sum_{j=k+1}^{k+\bar{k}}\left(\frac{\lambda_k-\lambda_j}{s_j}\right)\alpha_j^2,\end{aligned}\tag{2.26}$$

thus

$$I(u)=\frac{1}{2}\left(\|u^+\|_*^2-\|u^-\|_*^2\right)-\int_0^L F(t,u)dt,\tag{2.27}$$

$$I'(u)w=\langle u^+,\ w\rangle_*-\langle u^-,\ w\rangle_*-\int_0^L f(t,u)wdt,\qquad \forall u,\ w\in X.\tag{2.28}$$

Case (ii): $\mu = \lambda_k = \min\{\lambda_j\}_{j\geq 1}$. Under this assumption, for $\forall u \in X, u = \sum_{j=1}^{\infty} \alpha_j v_j$, we also have

$$I_1(u) = \frac{1}{2}\int_0^L \left[\left| \left(\sum_{j=1}^{\infty} \alpha_j v_j \right)'' \right|^2 - a \left| \left(\sum_{j=1}^{\infty} \alpha_j v_j \right)' \right|^2 - \lambda_k \left(\sum_{j=1}^{\infty} \alpha_j |v_j| \right)^2 \right] dt$$
$$= \sum_{j=1}^{\infty} \left[\alpha_j^2 \int_0^L |v_j''|^2 dt - a\alpha_j^2 \int_0^L |v_j'|^2 dt - \lambda_k \alpha_j^2 \int_0^L |v_j|^2 dt \right]$$
$$= \sum_{j=1}^{\infty} \left(\frac{\lambda_j - \lambda_k}{s_j} \right) \alpha_j^2. \tag{2.29}$$

Let

$$u^+ = \sum_{j\neq k}^{\infty} \alpha_j v_j, \qquad u^0 = \alpha_k v_k, \tag{2.30}$$

and

$$X^+ = Span\{v_j \mid j \neq k\}, \qquad X^0 = Span\{v_k\}, \tag{2.31}$$

then we conclude

$$u = u^+ + u^0, \qquad u^+ \in X^+, \qquad u^0 \in X^0, \qquad X = X^+ \oplus X^0, \tag{2.32}$$

and

$$\left(\frac{\lambda_1 - \lambda_k}{s_1} \right) \sum_{j=1}^{k-1} \alpha_j^2 + \sum_{j=k+1}^{\infty} \alpha_j^2 \geq \sum_{j\neq k} \left(\frac{\lambda_j - \lambda_k}{s_j} \right) \alpha_j^2$$
$$\geq \left(\frac{\lambda_{k-1} - \lambda_k}{s_{k-1}} \right) \sum_{j=1}^{k-1} \alpha_j^2 + \left(\frac{\lambda_{k+1} - \lambda_k}{s_{k+1}} \right) \sum_{j=k+1}^{\infty} \alpha_j^2, \tag{2.33}$$

which implies a new norm $\|\cdot\|_*$ as follows:

$$\|u\|_*^2 = \sum_{j\neq k} \left(\frac{\lambda_j - \lambda_k}{s_j} \right) \alpha_j^2 + \alpha_k^2. \tag{2.34}$$

equivalent with $\|\cdot\|$. Obviously, one obtains

$$\|u^+\|_*^2 = \sum_{j\neq k} \left(\frac{\lambda_j - \lambda_k}{s_j} \right) \alpha_j^2. \tag{2.35}$$

Consequently, we also have the same results as (2.27)–(2.28) with $u^- = 0$. □

Lemma 2.2. *Under the assumptions of Theorem 1.2, the functional $I(u)$ defined in (2.6) satisfies the Palais–Smale condition.*

Proof. Assume that $\{u_m\} \subset X$ satisfy

$$|I(u_m)| \leq C_0, \qquad I'(u_m) \to 0. \tag{2.36}$$

Writing $u_m = u_m^+ + u_m^- + u_m^0$, $u_m^+ \in X^+$, $u_m^- \in X^-$, $u_m^0 \in X^0$, then, for m sufficiently large, one has

$$|I'(u_m)u_m^+| = \langle u_m^+, u_m^+ \rangle_* - \langle u_m\ ,\ u_m^+ \rangle_* - \int_0^L f(t, u_m^+) u_m^+ dx$$
$$= \|u_m^+\|_*^2 - \int_0^L f(t, u_m^+) u_m^+ dt \leq \|u_m^+\|_*, \tag{2.37}$$

Using $(f2)$ and the Sobolev embedding inequality, we obtain

$$\left|\int_0^L f(t,u_m^+)u_m^+dt\right| \le K_1\int_0^L |u_m^+|^{\alpha+1}dt + K_2\int_0^L |u_m^+|dt$$
$$\le C_1(\|u_m^+\|_* + \|u_m^+\|_*^{\alpha+1}), \tag{2.38}$$

with some constant $C_1 > 0$. Combining (2.37) with (2.38) derives

$$\|u_m^+\|_*^2 \le (1+C_1)\|u_m^+\|_* + C_1\|u_m^+\|_*^{\alpha+1}, \tag{2.39}$$

thus we reduce that $\{u_m^+\}$ is bounded since $1 \le \alpha+1 < 2$. In the same way, $\{u_m^-\}$ is also bounded. Therefore there is a $C_2 > 0$ such that

$$\|u_m - u_m^0\|_* = \|u_m^+ + u_m^-\|_* \le C_2, \qquad \forall m \ge 1. \tag{2.40}$$

Next, with the aid of $(f2)$ and the mean value theorem, we easily show

$$\begin{aligned}
&\left|\int_0^L [F(t,u_m) - F(t,u_m^0)]dx\right| \\
&\le \int_0^L \left|f(t,u_m^0 + \xi(u_m - u_m^0))(u_m - u_m^0)\right| dt \ (0<\xi<1) \\
&\le C_3\left(\int_0^L |(u_m - u_m^0)|\,|u_m^0|^\alpha\,dt + \int_0^L |(u_m - u_m^0)|^{\alpha+1}dt + \int_0^L |(u_m - u_m^0)|\,dt\right) \\
&\le C_3\left(\|u_m^0\|_\infty^\alpha \int_0^L |(u_m - u_m^0)|dt + \int_0^L |(u_m - u_m^0)|^{\alpha+1}dt + \int_0^L |(u_m - u_m^0)|dt\right) \\
&\le C_4(\|u_m^0\|_\infty^\alpha \|u_m - u_m^0\|_* + \|u_m - u_m^0\|_*^{\alpha+1} + \|u_m - u_m^0\|_*)
\end{aligned} \tag{2.41}$$

for some constants $C_3, C_4 > 0$. Since $\dim X^0 = 1$, $u_m^0 \in X^0$, we know that $\|u_m^0\|_\infty$ is equivalent with $\|u_m^0\|$, it is easy to conclude by (2.40) and (2.41)

$$\left|\int_0^L \left[F(t,u_m) - F(t,u_m^0)\right] dx\right| \le C_5\|u_m^0\|^\alpha + C_6 \tag{2.42}$$

for some constants $C_5, C_6 > 0$. By Lemma 2.1, $I(u_m)$ can be written as

$$c_0 \ge I(u_m) = \frac{1}{2}\left(\|u_m^+\|_*^2 - \|u_m^-\|_*^2\right) - \int_0^L \left[F(t,u_m) - F(t,u_m^0)\right] dt - \int_0^L F(t,u_m^0)dt. \tag{2.43}$$

From (2.42) and (2.43), we obtain

$$C_5\|u_m^0\|^\alpha + \int_0^L F(t,u_m^0)dt \ge C_7 \tag{2.44}$$

with some constant C_7. If $\{u_m^0\}$ is unbounded, then

$$\liminf_{m\to\infty} \int_0^L F(t,u_m^0)dt/\|u_m^0\|^\alpha \ge -C_5. \tag{2.45}$$

We note that for $\forall m \ge 1$, u_m^0 can be expressed by $u_m^0 = c_m^0 \sin\frac{k\pi t}{L}, c_m^0 \in \mathbb{R}$, so $\|u_m^0\|^\alpha = |c_m^0|^\alpha(\frac{L}{2}s_k)^{\alpha/2}$. Here, k is fixed, consequently, (2.45) contradicts $(f4)$. Thus, we conclude that $\{u_m\}$ is bounded on X, and, using the standard method, $\{u_m\}$ has a convergent subsequence. □

Lemma 2.3. *Under the assumptions of Theorem 1.2, the functional $I(u)$ defined in (2.6) is bounded from below on X^+.*

Proof. According to $(f2)$, for any $u \in X$, we have

$$\left| \int_0^L F(t,u)dt \right| \leq \frac{K_1}{\alpha+1} \int_0^L |u|^{\alpha+1} dt + K_2 \int_0^L |u| dt \leq C_8(\|u\|_*^{\alpha+1} + \|u\|_*) \tag{2.46}$$

with some constant $C_8 > 0$. In particular, if $u \in X^+$, then for $\|u\|_* \to \infty$, we get

$$I(u) = \frac{1}{2}\|u\|_*^2 - \int_0^L F(t,u)dt \geq \frac{1}{2}\|u\|_*^2 - C_8(\|u\|_*^{\alpha+1} + \|u\|_*) \to \infty\ . \tag{2.47}$$

So I is bounded from below on X^+. □

3 Proof of the theorems

3.1 Proof of Theorem 1.2

Proof. In this section, we shall prove Theorem 1.2 by Theorem 1.5. Define

$$V = Span\{v_1, v_2, \ldots, v_{\hat{k}+\hat{l}+\hat{p}}\}, \tag{3.1}$$

$$Z = S_{\tilde{\rho}} \cap V = \left\{ u = \sum_{j=1}^{\hat{k}+\hat{l}+\hat{p}} \alpha_j v_j, \alpha_1, \alpha_2, \ldots, \alpha_{\hat{k}+\hat{l}+\hat{p}} \in \mathbb{R}, \sum_{j=1}^{\hat{k}+\hat{l}+\hat{p}} \alpha_j^2 = \tilde{\rho}^2 \right\} \tag{3.2}$$

with $\tilde{\rho} = \sqrt{\frac{Ls_1}{2(\hat{k}+\hat{l}+\hat{p})}}\rho$. For each $u \in Z$, according to (3.1), we have

$$u(t) = \sum_{j=1}^{\hat{k}+\hat{l}+\hat{p}} \alpha_j v_j = \sum_{j=1}^{\hat{k}+\hat{l}+\hat{p}} \alpha_j \sqrt{\frac{2}{Ls_j}} \sin\frac{j\pi t}{L}. \tag{3.3}$$

By the Cauchy–Schwarz inequality, one shows

$$\begin{aligned} |u(t)|^2 &\leq \frac{2}{L}\left(\sum_{j=1}^{\hat{k}+\hat{l}+\hat{p}} \frac{\alpha_j^2}{s_j}\right)\left(\sum_{j=1}^{\hat{k}+\hat{l}+\hat{p}} \sin^2\frac{j\pi t}{L}\right) \\ &\leq \frac{2}{L}\frac{2(\hat{k}+\hat{l}+\hat{p})}{Ls_1}\sum_{j=1}^{\hat{k}+\hat{l}+\hat{p}} \alpha_j^2 \\ &= \frac{2(\hat{k}+\hat{l}+\hat{p})}{Ls_1}\tilde{\rho}^2 = \rho^2. \end{aligned} \tag{3.4}$$

Therefore $|u(t)| \leq \rho$, $\forall t \in [0,L]$.

Thus, condition $(f3)$ implies that

$$
\begin{aligned}
I(u) &= \frac{1}{2}\int_0^L \left[|u''(t)|^2 - a|u'(t)|^2 - \lambda_k|u|^2\right] dt - \int_0^L F(t,u)dt \\
&\le \frac{1}{2}\int_0^L \left[|u''(t)|^2 - a|u'(t)|^2 - \lambda_k|u|^2\right] dt - \int_0^L \frac{M}{2}|u|^2 dt \\
&= \frac{1}{2}\sum_{j=1}^{\hat{k}+\hat{l}+\hat{p}} \alpha_j^2 \cdot \frac{2}{Ls_j}\left[\left(\frac{j\pi}{L}\right)^4 \int_0^L \left(\sin\frac{j\pi t}{L}\right)^2 dt - a\left(\frac{j\pi}{L}\right)^2 \int_0^L \left(\cos\frac{j\pi t}{L}\right)^2 dt\right] \\
&\quad - \frac{\lambda_k + M}{2}\int_0^L \sum_{j=1}^{\hat{k}+\hat{l}+\hat{p}} \alpha_j^2 \frac{2}{Ls_j}\left(\sin\frac{j\pi t}{L}\right)^2 dt \\
&= \frac{1}{2}\sum_{j=1}^{\hat{k}+\hat{l}+\hat{p}} \frac{\alpha_j^2}{s_j}\left[\left(\frac{j\pi}{L}\right)^4 - a\left(\frac{j\pi}{L}\right)^2 - (\lambda_k + M)\right].
\end{aligned}
\tag{3.5}
$$

Again, in view of $(f3)$, $\lambda_{\hat{k}+\hat{l}+\hat{p}} < \lambda_k + M$, and the property of the eigenvalue sequence $\{\lambda_k\}$, we have, for $1 \le j \le \hat{k} + \hat{l} + \hat{p}$,

$$
\left(\frac{j\pi}{L}\right)^4 - a\left(\frac{j\pi}{L}\right)^2 < (\lambda_k + M),
\tag{3.6}
$$

hence

$$
I(u) \le \frac{1}{2}\sum_{j=1}^{\hat{k}+\hat{l}+\hat{p}} \frac{\alpha_j^2}{s_j}\left[\left(\frac{j\pi}{L}\right)^4 - a\left(\frac{j\pi}{L}\right)^2 - (\lambda_k + M)\right] < 0.
\tag{3.7}
$$

Therefore, we get $\sup\{l(u) : u \in Z\} < 0$.

The functional (2.6) satisfies all hypotheses of Theorem 1.5, therefore it has at least $(\hat{k} + \hat{l} + \hat{p}) - (\bar{k} + 1)$ distinct pairs $(u_j, -u_j)$ of critical points. Since $I(u_j) < 0$ for $1 \le j \le (\hat{k} + \hat{l} + \hat{p}) - (\bar{k} + 1)$, we get $u_j \ne 0$ for $1 \le j \le (\hat{k} + \hat{l} + \hat{p}) - (\bar{k} + 1)$. □

3.2 Proof of Corollary 1.3

Proof. We first demonstrate the following result similar to Lemma 2.1. For $\lambda_k < \mu < \lambda_{k+1}$, $u = \sum_{j=1}^{\infty} \alpha_j v_j$, one has

$$
\begin{aligned}
I_1(u) &= \int_0^L \left[|u''(t)|^2 - a|u'(t)|^2 - \mu|u|^2\right] dt \\
&= \int_0^L \left[\left|\left(\sum_{j=1}^{\infty} \alpha_j v_j\right)''\right|^2 - a\left|\left(\sum_{j=1}^{\infty} \alpha_j v_j\right)'\right|^2 - \mu\left(\sum_{j=1}^{\infty} \alpha_j |v_j|\right)^2\right] dt \\
&= \sum_{j=1}^{\infty}\left[\alpha_j^2 \int_0^L |v_j''|^2 dt - a\alpha_j^2 \int_0^L |v_j'|^2 dt - \mu\alpha_j^2 \int_0^L |v_j|^2 dt\right] \\
&= \sum_{j=1}^{\infty}\left(\frac{\lambda_j}{s_j}\alpha_j^2 - \frac{\mu}{s_j}\alpha_j^2\right) \\
&= \sum_{j=1}^{k}\left(\frac{\lambda_j - \mu}{s_j}\right)\alpha_j^2 + \sum_{j=k+\bar{k}+1}^{\infty}\left(\frac{\lambda_j - \mu}{S_j}\right)\alpha_j^2 - \sum_{j=k+1}^{k+\bar{k}}\left(\frac{\mu - \lambda_j}{s_j}\right)\alpha_j^2.
\end{aligned}
\tag{3.8}
$$

Let

$$u^+ = \sum_{j=1}^{k} \alpha_j v_j + \sum_{j=k+\bar{k}+1}^{\infty} \alpha_j v_j, \qquad u^- = \sum_{j=k+1}^{k+\bar{k}} \alpha_j v_j, \tag{3.9}$$

$$\begin{aligned} X^+ &= Span\{v_j \mid 1 \le j \le k, \text{ or } j \ge k+\bar{k}+1\}, \\ X^- &= Span\{v_j \mid k+1 \le j \le k+\bar{k}\}, \end{aligned} \tag{3.10}$$

then $u = u^+ + u^-$, $u^+ \in X^+$, $u^- \in X^-$, $X = X^+ \oplus X^-$, and $I(u) = \frac{1}{2}(\|u^+\|_*^2 - \|u^-\|_*^2) - \int_0^L F(t,u)dt$.

We can continue to follow the same ideas as in Lemma 2.1, Lemma 2.2, and Lemma 2.3 to complete the proof of Corollary 1.3. In addition, it is not hard to see that, for the present case of $\lambda_k < \mu < \lambda_{k+1}$, condition $(f4)$ appearing in Theorem 1.2 is not indispensable since $X^0 = 0$ in the orthogonal decomposition of X. The details should be left to the reader. □

Remark 3.1. For $\forall \beta \in (0, \frac{1}{2})$, $\gamma \in [0,1)$, we can take a function $H(s) \in \mathbb{C}^1([0,\infty),\mathbb{R})$ such that

$$s^{1+2\beta} \le H(s) \le s^{1+\beta}, \qquad \forall s \in [0,1], \tag{3.11}$$

$$-\frac{1}{2}s^\gamma \le H'(s) \le -\frac{1}{4}s^\gamma, \qquad \forall s \in [2,\infty). \tag{3.12}$$

Let $F(t,u) = H(|u|)[(\sin t)^{2j} + 2]$, $\forall j \ge 1$. Since $\lim_{u\to 0} F(t,u)/|u|^2 = \infty$ uniformly in $t \in \mathbb{R}$, straightforward estimates show that $F(t,u)$ satisfies $(f1)$–$(f4)$ in Theorem 1.2 with $\alpha \in [\gamma, 1)$.

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